Provably Total NP Search Problems

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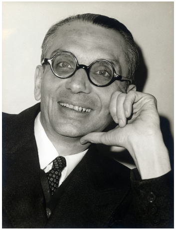
The Heritage

G. Peano (1858-1932)

A. Turing (1912-1954)



K. Gödel (1906-1978)

G. Gentzen (1909-1945)

Peano Arithmetic (PA)

- domain: $\mathbb{N}$
- language: 0, 1, $+$, $\cdot$, $\leqslant$
- axioms: defining equations

  schema of full induction for all formulas:

$$A(0) \ \wedge \ \forall x(A(x) \to A(x+1)) \ \to \ \forall x A(x)$$

Gödel's 2nd Incompleteness Theorem: $\mathsf{PA} \nvdash \mathsf{Con}_{\mathsf{PA}}$

Gentzen's Consistency Proof: $\mathsf{PA} + \mathrm{TI}(\epsilon_0) \vdash \mathsf{Con}_{\mathsf{PA}}$

Fragments of PA

- Laurie Kriby and Jeff Paris end of 70's:
  $I\Sigma_n$, that is, induction restricted to Σ_n formulas

$$\exists x_1 \forall x_2 \ldots Q x_n \underbrace{\varphi(x_1, \ldots, x_n)}_{\text{bounded quantifiers}}$$

- Charles Parsons beginning of 70's:
  provable recursive functions of $I\Sigma_1$ = *prim. rec. functions.*

- Wainer, Ono & Kadota 70's & '80:
  provable recursive functions of PA = $<\epsilon_0$*-rec. functions.*

Bounded Arithmetic

Cook 1975:
equational theory PV

Buss 1985:
aligned to PA

Zambella 1997, Cook-Nguyen 2010:
language with sorts for strings and indices

The Language of Bounded Arithmetic

- similar to PA
- domain: $\mathbb{N}$
- language: 0, 1, $+$, $\cdot$, $\leqslant$ plus $|.|$, $\#$, ...

$$|x| = \text{binary length of } x$$

$$x\#y = 2^{|x|\cdot|y|} \text{ polynomial growth rate}$$

- bounded formulas:

Σ^b_1 : $\exists x_1 \leqslant s_1 \ \forall y \leqslant |t| \ A(x_1, y)$ — $\mathsf{NP} = \Sigma^p_1$

Σ^b_2 : $\exists x_1 \leqslant s_1 \ \forall x_2 \leqslant s_2 \ \exists y \leqslant |t| \ A(x_1, x_2, y)$ — $\mathsf{NP}^{\mathsf{NP}} = \Sigma^p_2$

$\vdots$

s_1, s_2, t terms, A quantifier-free

Theories of Bounded Arithmetic

BASIC = set of open formulas defining non-logical symbols

Induction:

Σ^b_i-Ind : $\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x)$

Σ^b_i-LInd : $\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(|x|)$

where $\varphi \in \Sigma^b_i$

Theories:

$$\begin{array}{lcl} S^1_2 & = & \text{BASIC} + \Sigma^b_1\text{-LInd} \\ S^2_2 & = & \text{BASIC} + \Sigma^b_2\text{-LInd} \\ & \vdots & \\ S_2 & = & \bigcup_i S^i_2 \end{array} \qquad \begin{array}{lcl} T^1_2 & = & \text{BASIC} + \Sigma^b_1\text{-Ind} \\ T^2_2 & = & \text{BASIC} + \Sigma^b_2\text{-Ind} \\ & \vdots & \\ T_2 & = & \bigcup_i T^i_2 \end{array}$$

Definable Functions

Definition

f Σ_1^b-definable function in S_2^1 iff exists Σ_1^b-formula φ s.t.

- φ defines graph of f over $\mathbb{N}$, i.e. $f(x) = y \iff \mathbb{N} \models \varphi(x, y)$
- $S_2^1 \vdash \forall x\ \exists y \leqslant t(x)\ \varphi(x, y)$ for some term t
- $S_2^1 \vdash \forall x, y, y'\ (\varphi(x, y) \wedge \varphi(x, y') \rightarrow y = y')$

Theorem (Buss '85)

Σ_1^b-*definable functions in* S_2^1 = *p-time functions* FP

Other Characterisations

Theories	Induction	Graph Definability	Computational Complexity
S^1_2	Σ^b_1-LInd	Σ^b_1	FP
S^2_2	Σ^b_2-LInd	Σ^b_2	FP^{NP}
S^{k+1}_2	Σ^b_{k+1}-LInd	Σ^b_{k+1}	$FP^{\Sigma^p_k}$
U^1_2	$\Sigma^{1,b}_1$-LInd	$\Sigma^{1,b}_1$	FPSPACE
V^1_2	$\Sigma^{1,b}_1$-Ind	$\Sigma^{1,b}_1$	FEXPTIME

Results

Theorem (Buss'85, Buss'90)

$$S^1_2 \subseteq T^1_2 \preceq_{\forall\Sigma^b_2} S^2_2 \subseteq T^2_2 \preceq_{\forall\Sigma^b_3} S^3_2 \ \ldots$$

Main Open Problem

Is the hierarchy of theories strict?

Further Results

Independence Results [Krajíček, Pudlák, Takeuti '91, Krajíček'93, Jeřábek '09]

Separation of polynomial time hierarchy PH implies separation of BA theories.

Theorem [Buss '95, Zambella '96, Jeřábek '09]

Collapse of BA theories is equivalent to collapse of PH provable in BA

Propositional Proof Complexity [Cook, Reckhow '79]

Deep connections to propositional proof complexity.

Total NP Search Problems (TFNP)

Definition (Meggido-Papadimitriou '91; Papadimitriou '94)

A *Total NP Search Problem* (TFNP, for *total functional NP*) is a polynomial time computable (ptime), binary relation R, such that

- R is *honest*, i.e. polynomially bounded:
  If $R(x, y)$ then $|y| \leqslant p(|x|)$ for some polynomial p.
- R is *total*: For all x, there exists y s.t. $R(x, y)$.

The *search task* is:
Given input x, find a y s.t. $R(x, y)$.

TFNP intermediate between P and NP

Theorem

If $\mathsf{TFNP} \subseteq \mathrm{FP}$*, then* $\mathsf{NP} \cap \mathsf{coNP} = \mathsf{P}$.

Theorem

If $\mathsf{P} = \mathsf{NP}$*, then* $\mathsf{TFNP} \subseteq \mathrm{FP}$.

Search Complexity Classes in TFNP

Papadimitriou et.al. identified several natural sub-classes of TFNP.

defining principles: combinatorial lemma guaranteeing totality

- PPA *every graph has an even number of odd-degree nodes*
- PPP *there is no injective map from* [x] *to* [x−1]
- PLS *every directed acyclic graph has a sink*
- ...

In this talk: totality guaranteed by mathematical theories like Bounded Arithmetic or Peano Arithmetic.

Provable Total NP Search Problems

Let TFNP(T) be *set of provably total* NP *search problems in* T.

Consider

$$\begin{aligned} \mathrm{FP} = \mathsf{TFNP}(\mathrm{S}^1_2) &\subseteq \mathsf{TFNP}(\mathrm{T}^1_2) \subseteq \mathsf{TFNP}(\mathrm{T}^2_2) \subseteq \cdots \subseteq \mathsf{TFNP}(\mathrm{BA}) \\ &\subseteq \mathsf{TFNP}(\mathrm{I}\Sigma_1) \subseteq \cdots \subseteq \mathsf{TFNP}(\mathrm{PA}) \subseteq \mathsf{TFNP}(\mathrm{ZFC}) \end{aligned}$$

Is this hierarchy strict? Anywhere? (Implies $\mathsf{P} \neq \mathsf{NP}$!)

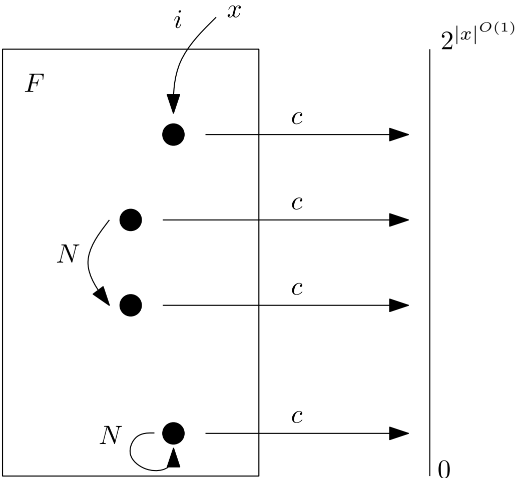
Π^b_k-Polynomial Local Search

Π^b_k-Polynomial Local Search

On input x:
F set of feasible solutions with polynomial bound d
i initial value
N neighbourhood function
c cost function

Search task: Find $s \in F(x)$ with $N(x, s) = s$.

N, i, c ptime, $F \in \Pi^b_k$



A Π^p_k-PLS *problem* $(k > 0)$ on instance x consists of,

- set of feasible solutions $F(x)$,
- neighbourhood function $N(x,s)$,
- cost function $c(x,s)$,
- initial feasible solution $i(x)$,

such that N, c, i polynomial time, F in Π^p_k, and

$$(\forall x,s)(s \in F(x) \to |s| \leqslant d(|x|)) \tag{1}$$

$$(\forall x)(i(x) \in F(x)) \tag{2}$$

$$(\forall x,s)(s \in F(x) \to N(x,s) \in F(x)) \tag{3}$$

$$(\forall x,s)(s \in F(x) \wedge N(x,s) \neq s \to c(x,N(x,s)) < c(x,s)) \tag{4}$$

Search task: find some s with $N(x,s) = s$.

Remark: Π^p_0-PLS, i.e. F ptime, is the same as PLS.

Theorem (Buss, Krajíček'95)

The provably total NP *search problems in* T^1_2 *are exactly characterised by Polynomial Local Search (PLS)*

Proposition

Let $L = (F, N, c, i)$ *be a* Π^b_k*-PLS problem. Then* $(\forall x)(\exists s)(N(x,s) = s)$ *is a provably total* NP *search problem in* T^{k+1}_2.

Proof.

Idea: Consider $\{d \colon (\exists s)(c(x,s) = d \wedge s \in F(x)\}$ $(\in \Sigma^b_{k+1})$.
Provable in T^{k+1}_2, we can determine minimum of this set. Choose some $s \in F(x)$ of minimal cost, this will satisfies $N(x,s) = s$. □

Theorem (B., Buss '09/'10)

Let $0 < k$. *The provably total* NP *search problems in* T^{k+1}_2 *are exactly characterised by* Π^b_k*-PLS problems.*

Other characterisations

CPLS

a version PLS similar to Π^b_1-PLS

Theorem (Kołodziejczyk, Skelly, Thapen '07)

$\mathsf{TFNP}(T^2_2)$ *characterised by CPLS*

k-turn Game Induction Principle, GI_k

k-turn two player games based on induction

Theorem (Skelly, Thapen '11)

$\mathsf{TFNP}(T^k_2)$ *characterised by* GI_k

Local Improvement Principles

Definition (k-round Local Improvement Principle LI_k)

Labels on a directed acyclic graph on $[x]$ can be updated in a consistent, well-founded way for k rounds.

LI (no subscript) allows $k = x$ (exponentially) many rounds

LLI – graph is a line

RLI – graph is a rectangle

Theory	Many-One Complete	
T^k_2	LI_k	[KNT'11]
V^1_2	LI	[KNT'11]
V^1_2	$\mathsf{LI}_{\log}$, LI with $O(\log n)$ may rounds	[BB'14]
U^1_2	LLI, Linear LI	[BB'14]
U^1_2	$\mathsf{LLI}_{\log}$	[KNT'11]
V^1_2	RLI, Rectangular LI	[KNT'11]
V^1_2	$\mathsf{RLI}_{\log}$	[BB'14]
U^1_2	RLI_1	[BB'14]

Provably Total NP Search Problems of Peano Arithmetic

$\prec$-bounded Local Search

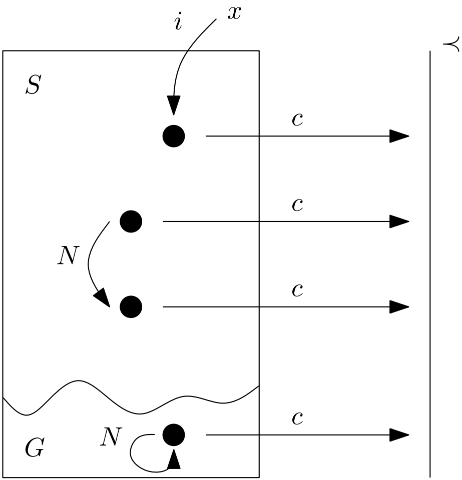
$\prec$ a ptime well-ordering.

S set of possible solutions
S need not be polynomially bounded!
i initial value
N neighbourhood function
c cost function

$$N(x,s) = s \vee c(N(x,s)) \prec c(s)$$

$$N(x,s) = s \;\Rightarrow\; |s| \leqslant d(|x|)$$

Search task:
Find s with $N(x,s) = s$.



A *$\prec$-bounded local search ($\prec$-bls) problem* on input x consists of a set $S(x)$ of possible solutions, a polynomial bound d, a neighbourhood function $\lambda s.N(x,s)\colon S(x) \to S(x)$, a cost function $\lambda s.c(x,s)\colon S(x) \to \mathbb{N}$, a function computing an initial solution $i(x)$,
such that $\prec, S, N, c, i$ are ptime, and

$$\prec \text{ is a total order} \tag{5}$$

$$(\forall x)(i(x) \in S(x)) \tag{6}$$

$$(\forall x, s)(s \in S(x) \to N(x,s) \in S(x)) \tag{7}$$

$$(\forall x, s)(N(x,s) = s \ \lor \ c(x, N(x,s)) \prec c(x,s)) \tag{8}$$

$$(\forall x, s)(N(x,s) = s \to |s| \leqslant d(|x|)) \tag{9}$$

Search task: find some s with $N(x,s) = s$.

Properties of $\prec$-bls Problems

Fact

Any $\prec$-bls problem defines a total NP search problem.

Remark

PLS = $<$-bls with polynomially bounded set of possible solutions.

Definition (Formalised $\prec$-bls Problems)

A $\prec$-bls problem is *formalised* provided the predicates S and $\prec$ are given by Δ^b_1-formulas, and the functions i, N, c are Σ^b_1-definable, such that the $\prec$-bls conditions are provable in S^1_2.

Theorem (B.'09)

The provably total NP search problems in PA are characterised by formalised α-bls problems, for $\alpha \prec_{\epsilon_0} \epsilon_0$.

Proof Idea

Notation for cut-elimination based on Mintz' continuous cut-elimination

Grigori E. Mints. Finite investigations of transfinite derivations. *Journal of Soviet Mathematics*, 10:548–596, 1978.

Wilfried Buchholz. Notation systems for infinitary derivations. *Archive for Math. Logic*, 30:277–296, 1991.

Klaus Aehlig and AB. On the computational complexity of cut-reduction. *APAL*, 161: 711–736, 2010.

A Notation System for (infinitary) propositional logic is a set $\mathcal{D}$ with functions

- for last inference, derived formula, and cut-rank
- $\mathtt{d}[\mathrm{j}]$: a *notation* of jth sub-derivation of (derivation denoted by) $\mathtt{d}$
- $\mathrm{o}(\mathtt{d})$: height of derivation tree (denoted by) $\mathtt{d}$.

AB, S.R.Buss, and C.Pollett. Ordinal Notations and Well-Orderings in Bounded Arithmetic. *APAL*, 120:197–223, 2003.

Further results

TFNP, propositional proof complexity, circuit complexity

- [Göös et.al. '22]
  Relate simulation between propositional proof systems to characterisations of TFNP classes, using this for separations of TFNP classes (relative to an oracle)
- [de Rezende, Göös, Robere '22; Buss, Fleming, Impagliazzo '23]
  TFNP as an organizing principle for connections between propositional proof systems and models of boolean circuits
- [Hubáček, Khaniki, Thapen '24]
  Connections between intersection classes in TFNP and proof properties (feasible disjunction property)

Open Questions / Future Work

- Turn TFNP characterisations into propositional tautologies and show lower bounds in related propositional proof systems for them.
- Separate T^2_2 from T^3_2 with $\forall\Sigma^b_1$ sentences (relativised)
- Analyse TFNP(T) for other theories.
  Conjecture: All theories which admit a good ordinal analysis also admit a characterisation of their provably total NP search problems, similar to PA.

Thanks!