Completeness in University Algebra: A New Perspective on High School Identities

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Abstract

We propose a new foundational setting for reasoning about isomorphisms between types and exponential identities, called University Algebra. Building on the failure of completeness in High School Algebra (HSA) and its categorification via bicartesian closed categories, we explore how categories with families (CwFs) equipped with Π-types, Σ-types, 1, and 2 can recover the missing isomorphisms. Notably, Wilkie's counterexample—which exposes the incompleteness of exponential semirings—becomes derivable in this framework using the type-theoretic axiom of choice. We conjecture that all definable isomorphisms in University Algebra can be reduced to equational ones and outline a syntactic strategy toward proving this completeness.

High School Algebra and Tarski's Conjecture. High School Algebra (HSA) is the equational theory of exponential semirings: algebraic structures with operations $+, \cdot, 0, 1$, and exponentiation satisfying natural identities such as distributivity and associativity. A classical question, going back to Tarski, asked whether all identities valid in all such models are derivable from a finite set of axioms — in other words, whether HSA is complete.

This was answered negatively by Wilkie [1] who exhibited an identity involving exponential polynomials which holds in all exponential semirings but is not derivable from the standard axioms. This demonstrated that HSA is *incomplete* as a purely equational theory.

Categorification and BiCCCs. Fiore and Di Cosmo [2] introduced a categorified view of this setting via *bicartesian closed categories* (biCCCs), categorifying the structure of exponential semirings: products model multiplication, coproducts model addition, and exponentials correspond to the exponential operation. In this setting, types and definable functions correspond to objects and morphisms in the biCCC.

This leads to a natural strengthening of the original question: *Are all isomorphisms between definable types derivable from the canonical equational properties of sums, products, and exponentials?* Wilkie's example shows that even in biCCCs, there are isomorphisms between definable types that are not compositionally derivable from the basic equational structure. This failure traces back to the lack of a genuine interaction law between sums and exponentials.

University Algebra. We propose to resolve this by replacing the biCCC framework with *University Algebra* (UA), modeled internally using *categories with families* (CwFs) equipped with:

- dependent function types (Π),
- dependent pair types (Σ),
- the terminal type (1), and
- a Boolean type (2), defined algebraically as an initial 2-algebra.

In this setting, the familiar isomorphisms of HSA are definable internally using only the universal properties of the type formers. Crucially, the type-theoretic axiom of choice — a derived principle in CwFs — allows us to prove isomorphisms like Wilkie's in a compositional manner. The dual nature of Σ-types, which can be viewed both as products (dependent pairs) and as sums (disjoint unions), plays a key role in mediating the interaction between sums and function spaces.

Wilkie's Counterexample. The correct counterexample from Wilkie's original paper involves the failure of the following identity to be derivable compositionally:

$$(A^y + B^y)^x (C^x + D^x)^y = (A^x + B^x)^y (C^y + D^y)^x$$

This identity holds in all exponential semirings but cannot be derived equationally. Its symmetry suggests that a semantic equality holds, but no equational derivation exists within the classical axioms of HSA. This reveals a fundamental incompleteness in the system.

Type-Theoretic Axiom of Choice and Derivation. In University Algebra, we can derive this isomorphism using the *type-theoretic axiom of choice*, which states that:

$$\Pi_{x:A}\Sigma_{y:B(x)}C(x,y) \cong \Sigma_{f:\Pi_{x:A}B(x)}\Pi_{x:A}C(x,f(x))$$

This axiom allows us to decompose and reassemble function types over dependent pairs in a way that captures the missing interaction. In particular, Wilkie's counterexample becomes derivable by interpreting exponentials via Π-types and distributing over Σ-types, exploiting the product-sum duality of Σ-types.

Conjecture and Proof Strategy. We conjecture that University Algebra is *complete*: every isomorphism between definable types is compositionally derivable from the universal properties of Π, Σ, 1, and 2.

Our proposed strategy is syntactic: to systematically study the space of definable isomorphisms using the internal language of CwFs, and to prove a Church-Rosser style property stating that every isomorphism can be reduced to a normal form built solely from canonical isomorphisms.

This approach unifies algebraic, categorical, and type-theoretic perspectives, and offers a new foundation for reasoning about type isomorphisms in functional languages and proof assistants.

References

[1] Alex J. Wilkie. On exponentiation – a solution to Tarski's high school algebra problem. *Connections Between Model Theory and Algebraic and Analytic Geometry*, 1991.

[2] Roberto Di Cosmo and Marcelo Fiore. Isomorphisms of generic programs. In *Proceedings of the 27th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL)*, 2000.