On Positivity of Exponential-Trigonometric Polynomials and Irrationality Exponents

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A (real) exponential-trigonometric polynomial (ETP) $f : \mathbb{R} \to \mathbb{R}$ is the solution of a homogeneous linear differential equation $c_n f^{(n)}(t) + c_{n-1} f^{(n-1)}(t) + \cdots + c_1 f'(t) + c_0 f(t) = 0$ with constant real coefficients $c_k \in \mathbb{R}$. Such functions arise in numerous applications in the mathematical sciences. An ETP is specified by an equation as above together with initial values $f^{(k)}(0) = u_k$, $k = 0, \dots, n-1$.

The *Positivity Problem* asks if an ETP is non-negative on the interval $[0, +\infty)$. This problem and its discrete-time analogue have received much attention for ETPs over the fields of rational or algebraic numbers, *i.e.* where the coefficients c_k and initial values u_k are rational or algebraic [3, 4, 7, 8, 9, 10, 11, 12]. It is a major open question whether this problem is decidable.

The focus on rational or algebraic numbers seems obvious from a computer science perspective, since these admit finite presentations with decidable equality. From a practical perspective, this restriction is not entirely natural: practitioners will likely want to be able to include special transcendental constants such as π in the specification of a function.

Hence, we are lead to the study of Positivity and related problems for ETPs over effectively presented countable fields that are obtained by the adjunction of finitely many transcendental numbers to the rationals. We begin with the following, somewhat curious, observation:

Proposition 1. Let $\alpha_1, \dots, \alpha_m$ be computable real numbers. Then the equality relation over the field $\mathbb{Q}(\alpha_1, \dots, \alpha_m)$ is decidable.

While Proposition 1 establishes the *existence* of an equality-testing algorithm, its proof relies on a non-constructive case distinction: Proceeding by induction on $m \geq 0$ and letting $K = \mathbb{Q}(\alpha_1, \dots, \alpha_{m-1})$, either the element α_m is transcendental over K — in which case equality-testing over $K(\alpha_m)$ trivially reduces to equality-testing over K — or α_m has a minimal polynomial over K, so that equality-testing over $K(\alpha_m)$ reduces to Euclidean division in the ring $K[X]$. We get an *explicit* equality-testing algorithm if and only if we fully understand all algebraic relations between the numbers $\alpha_1, \dots, \alpha_m$. For example, we know that there exists an algorithm for deciding equality over $\mathbb{Q}(e, \pi)$, but currently nobody knows what this algorithm looks like.

Many problems for linear differential equations or linear recurrence sequences are known to require the solution of very difficult Diophantine problems. Chonev, Ouaknine, and Worrell [5] have studied the problem of deciding whether a given ETP has infinitely many positive zeros. They showed that a decision method for this problem can be automatically translated into an algorithm to compute the Lagrange constant of any given real algebraic number – placing this problem well outside the reach of currently known techniques.

Our main result is a Diophantine hardness result for Positivity over fields as above in a similar spirit. We exhibit a very natural field over which the problem appears to be out of the scope of current techniques, and construct a much less natural field over which the problem is provably undecidable. Specifically, we relate decidability of Positivity to the computability of the *irrationality exponents* of the transcendental numbers we adjoin:

Definition 2. Let α be a real number. The irrationality exponent $\mu(\alpha)$ of α is the supremum of the set

$$\left\{\mu \in [0,+\infty) \mid \text{There exist infinitely many coprime integers } a, b \text{ with } 0 < \left|\alpha - \frac{a}{b}\right| < b^{-\mu}\right\}.$$

If the above set is unbounded, we let $\mu(\alpha) = +\infty$.

Rational numbers have irrationality exponent 1, while by Dirichlet's approximation theorem, irrational numbers have irrationality exponent at least 2. The Thue-Siegel-Roth theorem asserts that algebraic irrational numbers have irrationality exponent exactly 2. Numbers with infinite irrationality exponent are called *Liouville numbers*, following Liouville's construction of the first numbers known to be transcendental. In general, the irrationality exponent of a given number can be very difficult to find. For example, the irrationality exponent of π is only known to lie in the interval $[2, 5.096]$ (see also [1]). The following is our main result:

Theorem 3. Let $K \subseteq \mathbb{R}$ be an effectively presented countable field with decidable equality that effectively embeds into the real numbers. Assume that K contains the real number π. A decision method for the Positivity Problem over K can be automatically translated into an algorithm for approximating the irrationality exponent of any given member of K to any prescribed accuracy.

The reduction in Theorem 3 not only works for the Positivity Problem, but also for the Ultimate Positivity Problem, the Skolem Problem, and the Infinite Zero Problem (see [5] for definitions), making all these problems at least as hard as computing irrationality exponents.

Corollary 4. Consider the field $K = \mathbb{Q}(\pi)$. An algorithm for deciding Positivity of ETPs over K can be automatically translated into an algorithm for computing the irrationality exponent of π.

Becher, Bugeaud, and Slaman [2, Theorem 1] have shown that the irrationality exponents of computable numbers are precisely the number 1 and those numbers ≥ 2 which are upper limits of computable sequences of rational numbers. The latter are in general neither left- nor right-computably-enumerable, and thus a fortiori uncomputable. Proposition 1 immediately yields the existence of a field as in Theorem 3 over which Positivity becomes undecidable. However, as remarked earlier, we do not a priori know what the equality-decision method for this field looks like. By a slight modification of the construction in [2] we obtain a field with an *explicit* equality-testing algorithm over which Positivity becomes undecidable:

Theorem 5. There exists a computable number α which is algebraically independent of π such that $\mu(\alpha)$ is an uncomputable number. In particular, Positivity of ETPs over $\mathbb{Q}(\alpha, \pi)$ is undecidable.

Theorem 3 is proved by exhibiting an explicit family of ETPs whose eventual positivity is tied to the irrationality exponent of α, closely following the construction in [6, Example 2]. The key technical result is the following:

Lemma 6. Let $\alpha > 3/2$ be a real number. Let $\mu > 1$ be a real number with $\mu \neq \mu(\alpha)$. Consider the functions

$$h_{j,k}(x) = 2 + \cos\left(\tfrac{\pi}{\alpha}x + j\pi\right) + \cos(\pi x + k\pi) - x^{2-2\mu}$$

where $j, k \in \{0, 1\}$. Then $\mu > \mu(\alpha)$ if and only if all four functions are eventually positive, if and only if all four functions have finitely many zeros.

For non-integer μ, the functions $h_{j,k}$ in Lemma 6 are not exponential-trigonometric polynomials. However, for rational exponents $\mu = p/q$, the (eventual) positivity of $h_{j,k}$ is equivalent to that of the ETP

$$\left(2 + \cos\left(\tfrac{\pi}{\alpha}x + j\pi\right) + \cos(\pi x + k\pi)\right)^q - x^{2q-2p}.$$

Using this fact and Lemma 6 we can easily turn an algorithm for Positivity into an algorithm for computing $\mu(\alpha)$ to arbitrary accuracy by binary search.

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