Computability-theoretic properties of Hausdorff oracles

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Our intuitive understanding of "dimension" is that the dimension of a set (say, a subset of $\mathbb{R}^n$) ought to quantify *how* that set occupies its ambient space. This is distinct from "measure", which quantifies merely how large the set is. This was formalized for the first time by Hausdorff, who developed a notion of fractal dimension for subsets of $\mathbb{R}^n$.

Definition 0.1. [1] Let (X, ρ) be a metric space, let $E \subseteq X$, and let $\delta > 0$. Define

$$H^r_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(U_i)^r : E \subseteq \bigcup_{i=1}^{\infty} U_i \text{ and } \operatorname{diam}(U_i) \leq \delta \right\},$$

i.e. the infimum is taken over all countable covers $\mathcal{U} = \{U_1, U_2, \dots\}$ of E by open sets of diameter at most δ. The *r-dimensional Hausdorff measure* of E is

$$\mathcal{H}^r(E) = \lim_{\delta \to 0} H^r_\delta(E).$$

The *Hausdorff dimension* of E is

$$\dim_H(E) = \inf \{r : \mathcal{H}^r(E) = 0\} = \sup \{r : \mathcal{H}^r(E) = \infty\}.$$

Hausdorff dimension is useful for studying sets of Lebesgue measure 0, and it is one of the most well-studied aspects of geometric measure theory. Naïvely, one can effectivize this notion by requiring that the covers be computable; that is, an effective listing of balls with rational centers and rational radii.

Throughout the 21st century, various authors have contributed to the development of effective notions of fractal dimension. The most helpful characterization was by J. Lutz and Mayordomo, which was in terms of Kolmogorov complexity.

Definition 0.2. For $r \in \mathbb{N}$ and $x \in \mathbb{R}^n$, the *Kolmogorov complexity of x at precision r* is

$$K_r(x) = \min \{K(q) : q \in \mathbb{Q}^n \cap B_{2^{-r}}(x)\}.$$

Definition 0.3. [2] For $x \in \mathbb{R}^n$, the *effective dimension* of x relative to an oracle A is

$$\dim^A(x) = \liminf_{r \to \infty} \frac{K^A_r(x)}{r}.$$

The main breakthrough was the *point-to-set principle* of J. and N. Lutz, which states a direct correspondence between the classical fractal dimension of a set and the effective dimension of its points.

Theorem 0.4. [3] For all $E \subseteq \mathbb{R}^n$, we have

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x).$$

If A is such a minimizing oracle, we call A a *Hausdorff oracle* for E.

Several applications of this principle in geometric measure theory have been found, especially for improving lower bounds on the dimension of certain classes of sets. This is noteworthy, as it is uncommon for computability theory to have such immediate relevance to classical mathematics. However, as computability theorists, we are also interested in the computational power of Hausdorff oracles themselves.

In this talk, we will discuss results concerning this power. We consider the multivalued function $\mathbf{\Gamma} - \mathsf{HOracle}$ whose inputs are sets E in a pointclass $\mathbf{\Gamma}$ and which outputs a Hausdorff oracle for E. We show that $\mathbf{\Gamma} - \mathsf{HOracle}$ is Weihrauch equivalent to the problem $\mathbf{\Gamma} - \mathsf{HCover}$ of finding a *Hausdorff cover* of a set E, understood as a sequence of covers of E by basic open sets which witness its Hausdorff dimension as in the classical definition. This equivalence simplifies the investigation considerably.

We show that Hausdorff oracles (equivalently, Hausdorff covers) are, in general, exceptionally weak. For $\mathbf{\Sigma}^0_1$ and $\mathbf{\Pi}^0_1$ subsets of $\mathbb{R}^n$, Hausdorff oracles are computable outright, while Hausdorff oracles for $\mathbf{\Sigma}^0_2$ sets are uniformly computable from a name for the set in question. On the other hand, $\mathbf{\Pi}^0_2$ sets are non-computable but display strong cone-avoidance properties.

Theorem 0.5. Let $\mathbf{\Gamma}$ be a pointclass and $E \in \mathbf{\Gamma}(\mathbb{R}^n)$, and let $T \subseteq 2^{<\mathbb{N}}$ be a tree with no computable paths. There is $A \in \mathbf{\Gamma} - \mathsf{HOracle}(E)$ which does not compute any path on T.

Moreover, Hausdorff oracles for sets belonging to these higher pointclasses are very difficult to compute.

Theorem 0.6. $\mathbf{\Pi}^0_2 - \mathsf{HOracle} \not\leq_W \mathsf{UC}_{\omega^\omega}$.

We have the following as the best current upper bound:

Theorem 0.7. $\mathbf{\Pi}^0_2 - \mathsf{HOracle} \leq_W \mathbf{\Pi}^1_1 - \mathsf{C}_{\omega^\omega}$, i.e. choice on $\mathbf{\Pi}^1_1$ subsets of ω^ω.

From the perspective of the Weihrauch degrees, Hausdorff oracles are in a sense "off to the side". We also discuss briefly the computational difficulty of computing the dimension of sets of various pointclasses.

References

[1] Felix Hausdorff. Dimension und äusseres Mass. *Mathematische Annalen*, 79:157-179, 1919.

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